THE CALGARY MATHEMATICAL ASSOCIATION

37^{th} JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 24, 2013

NAME: ______________________________ GENDER: ☐ M ☐ F

PLEASE PRINT (First name Last name)

SCHOOL: ______________________________ GRADE: ________ (9,8,7,...)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART A has a total possible score of 45 points. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

MARKERS' USE ONLY	
PART A ________ $\times 5$	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.

THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

PART A: SHORT ANSWER QUESTIONS (Place answers in the boxes provided)

A1 From the set $\{1,2,3,4,5,6,7,8,9\}$, all odd numbers are removed. How many numbers are remaining?

A1

4

A2 A bag contains red, blue and green marbles. $2/3$ of the marbles are <u>not</u> red and $3/4$ of the marbles are <u>not</u> blue. What **fraction** of the marbles are <u>not</u> green? Express your fraction in **lowest terms**.

A2

$\frac{7}{12}$

A3 Ajooni walked 9 km at 4 km per hour, then biked for 4 hours at 9 km per hour. What was her average speed (in km per hour) for the entire trip?

A3

$\frac{36}{5} = 7.2$

A4 Notice that the digits of 2013 are four consecutive integers (because $0,1,2,3$ are consecutive integers). What was the last year (before 2013) whose digits were four consecutive integers?

A4

1432

A5 A circle is inscribed in an isosceles trapezoid, as shown, with parallel edges of lengths 8 and 18 cm and sloping edges of length L cm each. What is L?

A5

13

8

L L

18

A6 Mary has a large box of candies. If she gives a third of her candies to her mom, then a third of the remaining candies to her dad, and finally a third of what's left to her little sister, there will only be 16 candies in the box. How many candies are in the box at the beginning?

A6
54

A7 I have half a litre of solution, which is 40% acid, and the rest water. If I mix it with 2 litres of solution which is only 10% acid, what is the **percentage** of acid in the mixture?

A7
16

A8 A two-digit positive integer is said to be **doubly-divisible** if its two digits are different and non-zero, and it is exactly divisible by each of its two digits. For example, 12 is doubly-divisible since it is divisible by 1 and 2, whereas 99 is not doubly-divisible, since its digits are equal, and 90 is not doubly-divisible, because it contains a zero. What is the **largest** doubly-divisible positive integer?

A8
48

A9 What is the remainder when 2^{2013} is divided by 7?

A9
1

PART B: LONG ANSWER QUESTIONS

B1 You currently have \$100 and two magic wands A and B. Wand A increases the amount of money you have by 30% and wand B adds \$50 to the amount of money you have. You may use each wand exactly once, one after the other. In which order should you use the wands to maximize the amount of money you have? How much money would you have?

ANSWER If you first use wand A the \$100 becomes \$130, then applying wand B produces $\$130 + \$50 = \$180$.

However if you first use wand B you obtain \$150 which (after using wand A) becomes $\$150 \times 1.3 = \$195$.

So the maximum amount is \$195 obtained by using wand B first then wand A.

B2 Put one of the integers $1, 2, \ldots, 13$ into each of the boxes, so that twelve of these numbers are used once (and one number is not used at all), and so that all four equations are true. Be sure to explain how you found your answers.

$$\boxed{6} + \boxed{7} = \boxed{13}$$

$$\boxed{9} - \boxed{8} = \boxed{1}$$

$$\boxed{3} \times \boxed{4} = \boxed{12}$$

$$\boxed{10} \div \boxed{5} = \boxed{2}$$

One answer is shown above. The fourth equation (call it A divided by B equals C) is the same as B times C equals A, which is the same form as the 3rd equation. Neither of these equations can use the number 1 (or else there would be a repeated number), so in one of these two equations, the smallest number must be 2 and in the other the smallest number must be 3. If the smallest number is 3 the only possibility is 3 times 4 equals 12. This leaves 2 times 5 equals 10 as the only possibility for the other equation (since we cannot repeat the numbers 3 or 4). So the last two equations must use the six numbers 3,4,12,2,5,10, and there are various ways this can happen, for example we could have used 5 tmes 2 equals 10 and 12 divided by 4 equals 3 instead of what we wrote above.

Now the second equation (call it X minus Y equals Z) can be written as Z plus Y equals X, which is the same form as the first equation. So we need to find two equations of the form Z plus Y equals X using only numbers from 1,6,7,8,9,11,13. One of these equations cannot use the number 1 so it must be 6+7 =13. Then the only possibility for the other equation is 1+8=9. So the first two equations must use the numbers 6,7,13,1,8.9 in some order. The missing number must be 11.

B3 On planet X, an X-monkey has 2 legs and one head, while an X-hypercow has 3 legs and 4 heads. Robert has a herd of X-monkeys and X-hypercows on his farm, with a total of 87 legs and 86 heads in his herd. How many animals of each kind does Robert have?

ANSWER Let a be the number of X-monkeys and b the number of X-hypercows. Since X-monkeys have 2 legs, they contribute $2a$ legs to the total. There are $3b$ legs from the X-hypercows, for a total of 87 legs. This gives the equation

$$2a + 3b = 87.$$

Counting heads we get the equation

$$a + 4b = 86.$$

So we can write $a = 86 - 4b$ and substitute this into the first equation to get

$$2(86 - 4b) + 3b = 87$$

Then $5b = 85$ so $b = 17$ and then $a = 86 - 4 \times 17 = 18$. So the herd has 18 X-monkeys and 17 X-hypercows.

Another way to proceed is to note that taking one X-monkey and one X-hypercow gives 5 heads and 5 legs. Now $85 = 5 \times 17$, so 17 X-monkeys and 17 X-hypercows would give 85 heads and 85 legs. So you need 1 more head and 2 more legs, which is another X-monkey. So there are 18 X-monkeys and 17 X-hypercows.

B4 A pie is cut into a equal parts. Then one of these parts is cut into b smaller equal parts. Finally, one of the smaller parts is cut into c smallest equal parts. One of the original parts, together with a smaller part and a smallest part, makes up exactly three fifths of the pie. What are a, b and c (assuming a, b and c are integers greater than 1)?

ANSWER From the information the size of the first slices is $\frac{1}{a}$ of the whole pie, the size of the second slices is $\frac{1}{ab}$ of the whole pie, and the size of the third is $\frac{1}{abc}$ of the whole pie, so we have

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} = \frac{3}{5}.$$

Multiplying both sides by $5abc$ we get the equation

$$5bc + 5c + 5 = 3abc.$$

Now c is appears as a factor of all the summands except 5 so c must divide into 5 and since 5 is prime and c is bigger than 1, $c = 5$. Using this in the equation we obtain

$$25b + 25 + 5 = 15ab,$$

so $5b+6 = 3ab$. This tells us that b must divide into 6. Let us look at the possibilities. Trying $b = 2$ gives $10 + 6 = 6a$ which isn't possible since 6 doesn't divide into 16. Next try $b = 3$. This gives $15 + 6 = 9a$, which doesn't work since 9 doesn't divide into 21. So $b = 6$. Then $a = 2$.

The answer is $a = 2, b = 6, c = 5$.

B5 In a hockey tournament, five teams participated where each team played against each other team exactly once. A team receives 2 points for a win, 1 point for a tie and 0 points for a loss. At the end of the tournament the results showed that no two teams received the same total points, and the order of the teams (from highest point total to lowest point total) was A, B, C, D, E. Team B was the only team that did not lose any games and team E was the only team that did not win any games. How many points did each team receive and what was the result of each game?

ANSWER

	Total Points
A	6
B	5
C	4
D	3
E	2

	Winner (or tie)
A vs B	B
A vs C	A
A vs D	A
A vs E	A
B vs C	T
B vs D	T
B vs E	T
C vs D	C
C vs E	T
D vs E	D

Note that if we total the points for each match we obtain 2, so the point total recorded for the 10 games is 20. Since B was the only team that did not lose a game, A lost at least one game, making its maximum possible score 6. Its score could not be 5 since $5 + 4 + 3 + 2 + 1 = 15 < 20$. Thus Team A has three wins and one loss. Since B has no losses the game A lost must have been to B. Then B must have a tie in all three of its other games (otherwise, it has at least 6 points, at least as many as A). All the teams but E won some games, so both C and D won some game. Neither of them could win both games (excluding those with A and B about which we already know) because that would give a score of 5 which B got. Thus C won one of the other games and had one tie, and D won one game and had one loss, which means that E had one loss and one tie in the remaining games. Thus E lost the game with D and had a tie with C.

B6 The three edges of the base of a triangular pyramid (tetrahedron) each have length 6 units, and the height of the pyramid is 10. The other three (sloping) edges are equal in length. A sphere passes through all four corners of the pyramid. What is the radius of the sphere?

10

6

6 6

Note that triangle ABC is equilateral, so the medians of the three sides intersect at the centre O of the triangle. Let D be the midpoint of side AB, so that CD is a median of the triangle. Then triangle ADO is a right triangle. This triangle is similar to triangle CDA, so $OD : AD = AD : AC = 1 : 2$, and $AO = 2OD$. So, since $AO = CO$ you get $OD = 1/3CD$ Also $CD = \sqrt{6^2 - 3^2} = 3\sqrt{3}$. Then $OA = \frac{2}{3} \times 3\sqrt{3} = 2\sqrt{3}$.

Let V be the other vertex of the pyramid, and S the centre of the sphere. Then triangle SOA is a right-angled triangle, and if $h(= 10)$ is the height of the pyramid and $r$the radius of the sphere we get from Pythagoras' Theorem that

$$OA^2 = SA^2 - SO^2.$$

This becomes

$$(2\sqrt{3})^2 = r^2 - (h - r)^2$$

and

$$12 = 2rh - h^2 = 20r - 100.$$

Thus, $112 = 20r$, and the radius of the sphere is 5.6 cm.